2. Review of Vectors and Matrices

VECTORS

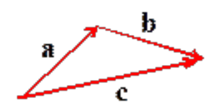
1. Definition

For the purposes of this course, a vector is an object which has magnitude and direction. Examples include forces, electric fields, and the normal to a surface. A vector is often represented pictorially as an arrow and symbolically by an underlined letter $\underline{a}$ or using bold type $\mathbf{a}$. Its magnitude is denoted $|\underline{a}|$ or $|\mathbf{a}|$. There are two special cases of vectors: the *unit vector* $\mathbf{n}$ has $|\mathbf{n}|=1$; and the *null vector* $\mathbf{0}$ has $|\mathbf{0}|=0$.

2. Vector Operations

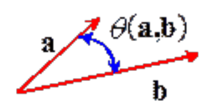
● **Addition**

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Then $\mathbf{c}=\mathbf{a}+\mathbf{b}$ is also a vector. The vector $\mathbf{c}$ may be shown diagramatically by placing arrows representing $\mathbf{a}$ and $\mathbf{b}$ head to tail, as shown in the figure.

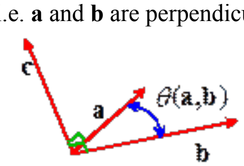


● **Multiplication**

1. **Multiplication by a scalar**. Let $\mathbf{a}$ be a vector, and α a scalar. Then $\mathbf{b}=\alpha\,\mathbf{a}$ is a vector. The direction of $\mathbf{b}$ is parallel to $\mathbf{a}$ and its magnitude is given by $|\mathbf{b}|=\alpha\,|\mathbf{a}|$.
   Note that you can form a unit vector $\mathbf{n}$ which is parallel to $\mathbf{a}$ by setting $\mathbf{n}=\frac{\mathbf{a}}{|\mathbf{a}|}$.
2. **Dot Product** (also called the scalar product). Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. The dot product of $\mathbf{a}$ and $\mathbf{b}$ is a scalar denoted by $\alpha=\mathbf{a}\cdot\mathbf{b}$, and is defined by
   $$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta(\mathbf{a},\mathbf{b}),$$
   where $\theta(\mathbf{a},\mathbf{b})$ is the angle subtended by $\mathbf{a}$ and $\mathbf{b}$. Note that $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$, and $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$. If $|\mathbf{a}|\neq 0$ and $|\mathbf{b}|\neq 0$ then $\mathbf{a}\cdot\mathbf{b}=0$ if and only if $\cos\theta(\mathbf{a},\mathbf{b})=0$; i.e. $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.



3. **Cross Product** (also called the vector product). Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector denoted by $\mathbf{c}=\mathbf{a}\times\mathbf{b}$. The direction of $\mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and is chosen so that (**a**,**b**,**c**) form a right handed triad, Fig. 3. The magnitude of $\mathbf{c}$ is given by
   $$|\mathbf{c}|=|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\,|\mathbf{b}|\sin\theta(\mathbf{a},\mathbf{b})$$
   Note that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ and $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=0$.



● **Some useful vector identities**

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$$

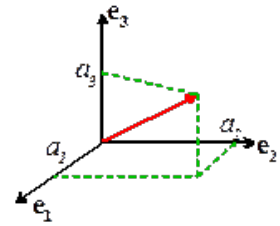
3. Cartesian components of vectors

Let $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ be three mutually perpendicular unit vectors which form a right handed triad, Fig. 4. Then $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are said to form and *orthonormal basis.* The vectors satisfy
$$|\mathbf{e}_1|=|\mathbf{e}_2|=|\mathbf{e}_3|=1 \qquad \mathbf{e}_1\times\mathbf{e}_2=\mathbf{e}_3, \quad \mathbf{e}_1\times\mathbf{e}_3=-\mathbf{e}_2 \quad \mathbf{e}_2\times\mathbf{e}_3=\mathbf{e}_1$$
We may express any vector $\mathbf{a}$ as a suitable combination of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. For example, we may write
$$\mathbf{a}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3=\sum_{i=1}^{3}a_i\mathbf{e}_i$$
where (a_1,a_2,a_3) are scalars, called the *components* of $\mathbf{a}$ in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. The components of $\mathbf{a}$ have a simple physical interpretation. For example, if we evaluate the dot product $\mathbf{a}\cdot\mathbf{e}_1$ we find that

$$\mathbf{a}\cdot\mathbf{e}_1=(a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3)\cdot\mathbf{e}_1=a_1$$

in view of the properties of the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Recall that

$$\mathbf{a}\cdot\mathbf{e}_1=|\mathbf{a}|\,|\mathbf{e}_1|\cos\theta(\mathbf{a},\mathbf{e}_1)$$



Then, noting that $|\mathbf{e}_1|=1$, we have

$$a_1=\mathbf{a}\cdot\mathbf{e}_1=|\mathbf{a}|\cos\theta(\mathbf{a},\mathbf{e}_1)$$

Thus, a_1 represents the projected length of the vector **a** in the direction of $\mathbf{e}_1$, as illustrated in the figure. Similarly, a_2 and a_3 may be shown to represent the projection of **a** in the directions $\mathbf{e}_2$ and $\mathbf{e}_3$, respectively.

The advantage of representing vectors in a Cartesian basis is that vector addition and multiplication can be expressed as simple operations on the components of the vectors. For example, let **a**, **b** and **c** be vectors, with components (a_1,a_2,a_3), (b_1,b_2,b_3) and (c_1,c_2,c_3), respectively. Then, it is straightforward to show that

$$\mathbf{c}=\mathbf{a}+\mathbf{b}\quad\Leftrightarrow\quad c_1=a_1+b_1;\quad c_2=a_2+b_2;\quad c_3=a_3+b_3$$

$$\mathbf{a}\cdot\mathbf{b}=\sum_{i=1}^{3}a_ib_i$$

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}\quad\Leftrightarrow\quad c_1=a_2b_3-a_3b_2;\quad c_2=a_3b_1-a_1b_3;\quad c_3=a_1b_2-a_2b_1$$

4. Change of basis

Let **a** be a vector, and let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be a Cartesian basis. Suppose that the components of **a** in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are known to be (a_1,a_2,a_3). Now, suppose that we wish to compute the components of **a** in a second Cartesian basis, $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$. This means we wish to find components $(\alpha_1,\alpha_2,\alpha_3)$, such that

$$\mathbf{a}=\alpha_1\mathbf{m}_1+\alpha_2\mathbf{m}_2+\alpha_3\mathbf{m}_3$$

To do so, note that

$$\alpha_1=\mathbf{a}\cdot\mathbf{m}_1=a_1\mathbf{e}_1\cdot\mathbf{m}_1+a_2\mathbf{e}_2\cdot\mathbf{m}_1+a_3\mathbf{e}_3\cdot\mathbf{m}_1$$
$$\alpha_2=\mathbf{a}\cdot\mathbf{m}_2=a_1\mathbf{e}_1\cdot\mathbf{m}_2+a_2\mathbf{e}_2\cdot\mathbf{m}_2+a_3\mathbf{e}_3\cdot\mathbf{m}_2$$
$$\alpha_3=\mathbf{a}\cdot\mathbf{m}_3=a_1\mathbf{e}_1\cdot\mathbf{m}_3+a_2\mathbf{e}_2\cdot\mathbf{m}_3+a_3\mathbf{e}_3\cdot\mathbf{m}_3$$

This transformation is conveniently written as a matrix operation

$$[\alpha]=[Q]\,[a],$$

where $[\alpha]$ is a matrix consisting of the components of **a** in the basis $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$, $[a]$ is a matrix consisting of the components of **a** in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, and $[Q]$ is a `rotation matrix' as follows

$$[\alpha]=\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}\qquad[a]=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\qquad[Q]=\begin{bmatrix}\mathbf{m}_1\cdot\mathbf{e}_1&\mathbf{m}_1\cdot\mathbf{e}_2&\mathbf{m}_1\cdot\mathbf{e}_3\\\mathbf{m}_2\cdot\mathbf{e}_1&\mathbf{m}_2\cdot\mathbf{e}_2&\mathbf{m}_2\cdot\mathbf{e}_3\\\mathbf{m}_3\cdot\mathbf{e}_1&\mathbf{m}_3\cdot\mathbf{e}_2&\mathbf{m}_3\cdot\mathbf{e}_3\end{bmatrix}$$

Note that the elements of $[Q]$ have a simple physical interpretation. For example, $\mathbf{m}_1\cdot\mathbf{e}_1=\cos\theta(\mathbf{m}_1,\mathbf{e}_1)$, where $\theta(\mathbf{m}_1,\mathbf{e}_1)$ is the angle between the $\mathbf{m}_1$ and $\mathbf{e}_1$ axes. Similarly $\mathbf{m}_1\cdot\mathbf{e}_2=\cos\theta(\mathbf{m}_1,\mathbf{e}_2)$ where $\theta(\mathbf{m}_1,\mathbf{e}_2)$ is the angle between the $\mathbf{m}_1$ and $\mathbf{e}_2$ axes. In practice, we usually know the angles between the axes that make up the two bases, so it is simplest to assemble the elements of $[Q]$ by putting the cosines of the known angles in the appropriate places.

Index notation provides another convenient way to write this transformation:

$$\alpha_i=Q_{ij}a_j,\qquad Q_{ij}=\mathbf{e}_i\cdot\mathbf{m}_j$$

You don't need to know index notation in detail to understand this – all you need to know is that

$$Q_{ij}a_j\equiv\sum_{j=1}^{3}Q_{ij}a_j$$

The same approach may be used to find an expression for a_i in terms of α_i. If you work through the details, you will find that

$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}\mathbf{m}_1\cdot\mathbf{e}_1&\mathbf{m}_2\cdot\mathbf{e}_1&\mathbf{m}_3\cdot\mathbf{e}_1\\\mathbf{m}_1\cdot\mathbf{e}_2&\mathbf{m}_2\cdot\mathbf{e}_2&\mathbf{m}_3\cdot\mathbf{e}_2\\\mathbf{m}_1\cdot\mathbf{e}_3&\mathbf{m}_2\cdot\mathbf{e}_3&\mathbf{m}_3\cdot\mathbf{e}_3\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$

Comparing this result with the formula for α_i in terms of a_i, we see that

$$[a] = [Q]^T [\alpha]$$

where the superscript T denotes the transpose (rows and columns interchanged). The transformation matrix $[Q]$ is therefore *orthogonal*, and satisfies

$$[Q]^{-1} = [Q]^T \qquad [Q][Q]^T = [Q]^T [Q] = [I]$$

where $[I]$ is the identity matrix.

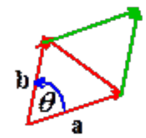
5. Useful vector operations

Calculating areas

The area of a triangle bounded by vectors **a**, **b** and **b-a** is

$$A = \frac{1}{2}\left|\mathbf{a} \times \mathbf{b}\right|$$

The area of the parallelogram shown in the picture is $2A$.



Calculating angles

The angle between two vectors **a** and **b** is

$$\theta = \cos^{-1}\left(\mathbf{a} \cdot \mathbf{b} / \left|\mathbf{a}\right| \left|\mathbf{b}\right|\right)$$

Calculating the normal to a surface.

If two vectors **a** and **b** can be found which are known to lie in the surface, then the unit normal to the surface is

$$\mathbf{n} = \pm \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

If the surface is specified by a parametric equation of the form $\mathbf{r} = \mathbf{r}(s, t)$, where s and t are two parameters and **r** is the position vector of a point on the surface, then two vectors which lie in the plane may be computed from

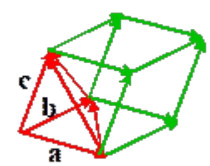
$$\mathbf{a} = \frac{\partial \mathbf{r}}{\partial s}, \qquad \mathbf{b} = \frac{\partial \mathbf{r}}{\partial t}$$

Calculating Volumes

The volume of the parallelopiped defined by three vectors **a**, **b**, **c** is

$$V = \left|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})\right|$$

The volume of the tetrahedron shown outlined in red is *V/6*.



VECTOR FIELDS AND VECTOR CALCULUS

1. Scalar field.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Let

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

denote the position vector of a point in space. A **scalar field** is a scalar valued function of position in space. A scalar field is a function of the components of the position vector, and so may be expressed as $\phi(x_1, x_2, x_3)$. The value of ϕ at a particular point in space must be independent of the choice of basis vectors. A scalar field may be a function of time (and possibly other parameters) as well as position in space.

2. Vector field

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Let

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

denote the position vector of a point in space. A **vector field** is a vector valued function of position in space. A vector field is a function of the components of the position vector, and so may be expressed as $\mathbf{v}(x_1, x_2, x_3)$. The vector may also be expressed as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

The magnitude and direction of **v** at a particular point in space is independent of the choice of basis vectors. A vector field may be a function of time (and possibly other parameters) as well as position in space.

3. Change of basis for scalar fields.

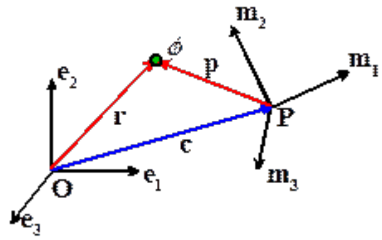
Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Express the position vector of a point relative to O in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

and let $\phi(x_1, x_2, x_3)$ be a scalar field.

Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a second Cartesian basis, with origin P. Let $\mathbf{c} \equiv \overrightarrow{OP}$ denote the position vector of P relative to O. Express the position vector of a point relative to P in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ as

$$\mathbf{p} = \xi_1\mathbf{m}_1 + \xi_2\mathbf{m}_2 + \xi_3\mathbf{m}_3$$



To find $\phi(\xi_1, \xi_2, \xi_3)$, use the following procedure. First, express $\mathbf{p}$ as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, using the procedure outlined in Section 1.4:

$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$$

where

$$\begin{aligned} p_1 &= \xi_1\mathbf{e}_1 \cdot \mathbf{m}_1 + \xi_2\mathbf{e}_2 \cdot \mathbf{m}_1 + \xi_3\mathbf{e}_3 \cdot \mathbf{m}_1 \\ p_2 &= \xi_1\mathbf{e}_1 \cdot \mathbf{m}_2 + \xi_2\mathbf{e}_2 \cdot \mathbf{m}_2 + \xi_3\mathbf{e}_3 \cdot \mathbf{m}_2 \\ p_3 &= \xi_1\mathbf{e}_1 \cdot \mathbf{m}_3 + \xi_2\mathbf{e}_2 \cdot \mathbf{m}_3 + \xi_3\mathbf{e}_3 \cdot \mathbf{m}_3 \end{aligned}$$

or, using index notation

$$p_i = Q_{ij}\xi_j$$

where the transformation matrix Q_{ij} is defined in Sect 1.4.

Now, express $\mathbf{c}$ as components in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and note that

$$\begin{aligned} &\mathbf{r} = \mathbf{p} + \mathbf{c} \\ &\Rightarrow \; x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3 + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \\ &\Rightarrow x_1 = p_1 + c_1, \quad x_2 = p_2 + c_2, \quad x_3 = p_3 + c_3 \\ &\Rightarrow x_i = Q_{ij}\xi_j + c_i \end{aligned}$$

so that

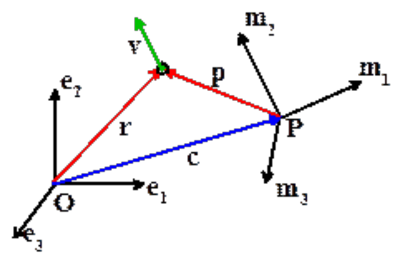
$$\begin{aligned} \phi(x_1, x_2, x_3) &= \phi(p_1 + c_1, p_2 + c_2, p_3 + c_3) \\ &= \phi(Q_{1j}\xi_j + c_1, Q_{2j}\xi_j + c_2, Q_{3j}\xi_j + c_3) \end{aligned}$$

4. Change of basis for vector fields.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Express the position vector of a point relative to O in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

and let $\mathbf{v}(x_1, x_2, x_3)$ be a vector field, with components



$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a second Cartesian basis, with origin P. Let $\mathbf{c} \equiv \overrightarrow{OP}$ denote the position vector of P relative to O. Express the position vector of a point relative to P in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ as

$$\mathbf{p} = \xi_1\mathbf{m}_1 + \xi_2\mathbf{m}_2 + \xi_3\mathbf{m}_3$$

To express the vector field as components in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ and as a function of the components of $\mathbf{p}$, use the following procedure. First, express (v_1, v_2, v_3) in terms of (ξ_1, ξ_2, ξ_3) using the procedure outlined for scalar fields in the preceding section

$$\begin{aligned} v_k(x_1, x_2, x_3) &= v_k(p_1 + c_1, p_2 + c_2, p_3 + c_3) \\ &= v_k(Q_{1j}\xi_j + c_1, Q_{2j}\xi_j + c_2, Q_{3j}\xi_j + c_3) \end{aligned}$$

for *k=1,2,3*. Now, find the components of $\mathbf{v}$ in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ using the procedure outlined in Section 1.4. Using index notation, the result is

$$\begin{aligned} \mathbf{v} = &Q_{1i}v_i(Q_{1j}\xi_j + c_1, Q_{2j}\xi_j + c_2, Q_{3j}\xi_j + c_3)\mathbf{e}_1 \\ &+ Q_{2i}v_i(Q_{1j}\xi_j + c_1, Q_{2j}\xi_j + c_2, Q_{3j}\xi_j + c_3)\mathbf{e}_2 \\ &+ Q_{2i}v_i(Q_{1j}\xi_j + c_1, Q_{2j}\xi_j + c_2, Q_{3j}\xi_j + c_3)\mathbf{e}_3 \end{aligned}$$

5. Time derivatives of vectors

Let **a***(t)* be a vector whose magnitude and direction vary with time, *t*. Suppose that $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is a *fixed* basis, i.e. independent of time. We may express **a***(t)* in terms of components (a_x, a_y, a_z) in the basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ as

$$\mathbf{a}(t) = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

The *time derivative* of **a** is defined using the usual rules of calculus

$$\dot{\mathbf{a}}(t) = \frac{d}{dt}\mathbf{a}(t) = \lim_{\in\to 0}\frac{\mathbf{a}(t+\in)-\mathbf{a}(t)}{\in},$$

or in component form as

$$\dot{\mathbf{a}}(t) = \dot{a}_x\mathbf{i} + \dot{a}_y\mathbf{j} + \dot{a}_z\mathbf{k}$$

The definition of the time derivative of a vector may be used to show the following rules

$$\frac{d}{dt}[\alpha(t)\mathbf{a}(t)] = \dot{\alpha}(t)\mathbf{a}(t) + \alpha(t)\dot{\mathbf{a}}(t)$$

$$\frac{d}{dt}[\mathbf{a}(t)\cdot\mathbf{b}(t)] = \dot{\mathbf{a}}(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\dot{\mathbf{b}}(t)$$

$$\frac{d}{dt}[\mathbf{a}(t)\times\mathbf{b}(t)] = \dot{\mathbf{a}}(t)\times\mathbf{b}(t) + \mathbf{a}(t)\times\dot{\mathbf{b}}(t)$$

6. Using a rotating basis

It is often convenient to express position vectors as components in a basis which rotates with time. To write equations of motion one must evaluate time derivatives of rotating vectors.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis which rotates with instantaneous angular velocity $\boldsymbol{\Omega}$. Then,

$$\frac{d\mathbf{e}_1}{dt} = \boldsymbol{\Omega}\times\mathbf{e}_1, \quad \frac{d\mathbf{e}_2}{dt} = \boldsymbol{\Omega}\times\mathbf{e}_2, \quad \frac{d\mathbf{e}_3}{dt} = \boldsymbol{\Omega}\times\mathbf{e}_3$$

7. Gradient of a scalar field.

Let ϕ be a scalar field in three dimensional space. The gradient of ϕ is a vector field denoted by $\mathrm{grad}(\phi)$ or $\nabla\phi$, and is defined so that

$$(\nabla\phi)\cdot\mathbf{a} = \lim_{\in\to 0}\frac{\phi(\mathbf{r}+\in\mathbf{a})-\phi(\mathbf{r})}{\in}$$

for every position **r** in space and for every vector **a**.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Let

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

denote the position vector of a point in space. Express ϕ as a function of the components of **r** $\phi = \phi(x_1, x_2, x_3)$. The gradient of ϕ in this basis is then given by

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3$$

8. Gradient of a vector field

Let **v** be a vector field in three dimensional space. The gradient of **v** is a tensor field denoted by $\mathrm{grad}(\mathbf{v})$ or $\nabla\mathbf{v}$, and is defined so that

$$(\nabla\mathbf{v})\cdot\mathbf{a} = \lim_{\in\to 0}\frac{\mathbf{v}(\mathbf{r}+\in\mathbf{a})-\mathbf{v}(\mathbf{r})}{\in}$$

for every position **r** in space and for every vector **a**.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Let

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

denote the position vector of a point in space. Express **v** as a function of the components of **r,** so that $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3)$. The gradient of **v** in this basis is then given by

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Alternatively, in index notation

$$[\nabla\mathbf{v}]_{ij} \equiv \frac{\partial v_i}{\partial x_j}$$

9. Divergence of a vector field

Let **v** be a vector field in three dimensional space. The divergence of **v** is a scalar field denoted by $\mathrm{div}(\mathbf{v})$ or $\nabla\cdot\mathbf{v}$. Formally, it is defined as $\mathrm{trace}(\mathrm{grad}(\mathbf{v}))$ (the trace of a tensor is the sum of its diagonal terms).

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Let
$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$
denote the position vector of a point in space. Express **v** as a function of the components of **r**: $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3)$. The divergence of **v** is then

$$\text{div}(\mathbf{v}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

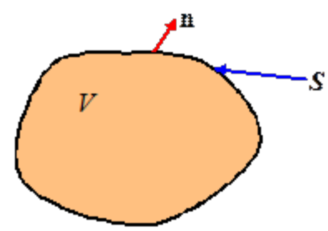
10. Curl of a vector field.

Let **v** be a vector field in three dimensional space. The curl of **v** is a vector field denoted by $\text{curl}(\mathbf{v})$ or $\nabla \times \mathbf{v}$. It is best defined in terms of its components in a given basis, although its magnitude and direction are not dependent on the choice of basis.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis with origin O in three dimensional space. Let
$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$
denote the position vector of a point in space. Express **v** as a function of the components of **r** $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3)$. The curl of **v** in this basis is then given by
$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3$$
Using index notation, this may be expressed as
$$[\nabla \times \mathbf{v}]_i = \in_{ijk} \frac{\partial v_k}{\partial x_j}$$

11. The Divergence Theorem.

Let V be a closed region in three dimensional space, bounded by an orientable surface S. Let **n** denote the unit vector normal to S, taken so that **n** points out of V. Let **u** be a vector field which is continuous and has continuous first partial derivatives in some domain containing T. Then
$$\int_V \text{div}(\mathbf{u})\, dV = \int_S \mathbf{u} \cdot \mathbf{n}\, dA$$
alternatively, expressed in index notation
$$\int_V \frac{\partial u_i}{\partial x_i} dV = \int_S u_i n_i dA$$



For a proof of this extremely useful theorem consult e.g. Kreyzig, *Advanced Engineering Mathematics,* Wiley, New York, (1998).

MATRICES

1. Definition

An $(n \times m)$ matrix $[A]$ is a set of numbers, arranged in *m* rows and *n* columns
$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- A **square matrix** has equal numbers of rows and columns
- A **diagonal matrix** is a square matrix with elements such that $a_{ij} = 0$ for $i \neq j$
- The **identity matrix** $[I]$ is a diagonal matrix for which all diagonal elements $a_{ii} = 1$
- A **symmetric matrix** is a square matrix with elements such that $a_{ij} = a_{ji}$
- A **skew symmetric matrix** is a square matrix with elements such that $a_{ij} = -a_{ji}$

2. Matrix operations

● **Addition** Let $[A]$ and $[B]$ be two matrices of order $(m \times n)$ with elements a_{ij} and b_{ij}. Then

$$[C] = [A] + [B] \Leftrightarrow c_{ij} = a_{ij} + b_{ij}$$

● **Multiplication by a scalar**. Let $[A]$ be a matrix with elements a_{ij}, and let k be a scalar. Then

$$[B] = k\,[A] \Leftrightarrow b_{ij} = k a_{ij}$$

● **Multiplication by a matrix.** Let $[A]$ be a matrix of order $(m \times n)$ with elements a_{ij}, and let $[B]$ be a matrix of order $(p \times q)$ with elements b_{ij}. The product $[C] = [A]\,[B]$ is defined only if n=p, and is an $(m \times q)$ matrix such that

$$[C] = [A]\,[B] \Leftrightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note that multiplication is distributive and associative, but not commutative, i.e.

$$[A]\,([B] + [C]) = [A]\,[B] + [A]\,[C] \qquad [A]\,([B]\,[C]) = ([A]\,[B])\,[C] \qquad [A]\,[B] \neq [B]\,[A]$$

The multiplication of a vector by a matrix is a particularly important operation. Let **b** and **c** be two vectors with n components, which we think of as $(1 \times n)$ matrices. Let $[A]$ be an $(m \times n)$ matrix. Thus

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \qquad [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Now,

$$\mathbf{c} = [A]\,\mathbf{b} \;\Leftrightarrow\; c_i = \sum_{j=1}^{n} a_{ij} b_j$$

i.e.

$$\begin{aligned} c_1 &= a_{11}b_1 + a_{12}b_2 + a_{13}b_3 + \cdots a_{1n}b_n \\ c_2 &= a_{21}b_1 + a_{22}b_2 + a_{23}b_3 + \cdots a_{2n}b_n \\ &\vdots \\ c_m &= a_{m1}b_1 + a_{m2}b_2 + a_{m3}b_3 + \cdots a_{mn}b_n \end{aligned}$$

● **Transpose.** Let $[A]$ be a matrix of order $(m \times n)$ with elements a_{ij}. The transpose of $[A]$ is denoted $[A]^T$. If $[B]$ is an $(n \times m)$ matrix such that $[B] = [A]^T$, then $b_{ij} = a_{ji}$, i.e.

$$[A]^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & a_{3m} & \cdots & a_{nm} \end{bmatrix}$$

Note that

$$([A]\,[B])^T = [B]^T [A]^T$$

● **Determinant** The determinant is defined only for a square matrix. Let $[A]$ be a (2×2) matrix with components a_{ij}. The determinant of $[A]$ is denoted by det $[A]$ or $|A|$ and is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Now, let $[A]$ be an $(n \times n)$ matrix. Define the *minors* M_{ij} of $[A]$ as the determinant formed by omitting the ith row and jth column of $[A]$. For example, the minors M_{11} and M_{12} for a (3×3) matrix are computed as follows. Let

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23} \qquad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$$

Define the *cofactors* C_{ij} of $[A]$ as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Then, the determinant of the $(n \times n)$ matrix $[A]$ is computed as follows

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$$

The result is the same whichever row *i* is chosen for the expansion. For the particular case of a (3×3) matrix

$$\det[A] = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

The determinant may also be evaluated by summing over rows, i.e.

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$$

and as before the result is the same for each choice of column *j*. Finally, note that

$$\det[A]^T = \det[A] \qquad \det([A][B]) = \det[A]\det[B]$$

● **Inversion.** Let $[A]$ be an $(n \times n)$ matrix. The inverse of $[A]$ is denoted by $[A]^{-1}$ and is defined such that

$$[A]^{-1}[A] = [I]$$

The inverse of $[A]$ exists if and only if $\det[A] \neq 0$. A matrix which has no inverse is said to be *singular*. The inverse of a matrix may be computed explicitly, by forming the *cofactor matrix* $[C]$ with components c_{ij} as defined in the preceding section. Then

$$[A]^{-1} = \frac{1}{\det[A]}[C]^T$$

In practice, it is faster to compute the inverse of a matrix using methods such as Gaussian elimination.

Note that

$$([A][B])^{-1} = [B]^{-1}[A]^{-1}$$

For a *diagonal matrix,* the inverse is

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$$

For a (2×2) matrix, the inverse is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

● **Eigenvalues and eigenvectors.** Let $[A]$ be an $(n \times n)$ matrix, with coefficients a_{ij}. Consider the vector equation

$$[A]\mathbf{x} = \lambda \mathbf{x} \qquad (1)$$

where **x** is a vector with *n* components, and λ is a scalar (which may be complex). The *n* nonzero vectors **x** and corresponding scalars λ which satisfy this equation are the *eigenvectors* and *eigenvalues* of $[A]$.

Formally, eighenvalues and eigenvectors may be computed as follows. Rearrange the preceding equation to

$$([A] - \lambda[I])\mathbf{x} = \mathbf{0} \qquad (2)$$

This has nontrivial solutions for **x** only if the determinant of the matrix $([A] - \lambda[I])$ vanishes. The equation

$$\det([A] - \lambda[I]) = 0$$

is an *n*th order polynomial which may be solved for λ. In general the polynomial will have *n* roots, which may be complex. The eigenvectors may then be computed using equation (2). For example, a (2×2) matrix generally has two eigenvectors, which satisfy

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

Solve the quadratic equation to see that

$$\lambda_1 = \frac{1}{2}(a_{11} + a_{22}) - \frac{1}{2}\left\{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})\right\}^{1/2}$$

$$\lambda_2 = \frac{1}{2}(a_{11} + a_{22}) + \frac{1}{2}\left\{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})\right\}^{1/2}$$

The two corresponding eigenvectors may be computed from (2), which shows that

$$\begin{bmatrix} a_{11} - \lambda_i & a_{12} \\ a_{21} & a_{22} - \lambda_i \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \end{bmatrix} = 0$$

so that, multiplying out the first row of the matrix (you can use the second row too, if you wish – since we chose λ to make the determinant of the matrix vanish, the two equations have the same solutions. In fact, if $a_{12} = 0$, you will need to do this, because the first equation will simply give 0=0 when trying to solve for one of the eigenvectors)

$$\left(\frac{1}{2}(a_{11}-a_{22})+\frac{1}{2}\left\{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})\right\}^{1/2}\right)x_1^{(1)}+a_{12}x_2^{(1)}=0$$
$$\left(\frac{1}{2}(a_{11}-a_{22})-\frac{1}{2}\left\{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})\right\}^{1/2}\right)x_1^{(2)}+a_{12}x_2^{(2)}=0$$

which are satisfied by any vector of the form

$$\mathbf{x}^{(1)}=\begin{bmatrix}2a_{12}\\ (a_{11}-a_{22})+\left\{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})\right\}^{1/2}\end{bmatrix}p$$
$$\mathbf{x}^{(2)}=\begin{bmatrix}2a_{12}\\ (a_{11}-a_{22})-\left\{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})\right\}^{1/2}\end{bmatrix}q$$

where *p* and *q* are arbitrary real numbers.

It is often convenient to *normalize* eigenvectors so that they have unit `length'. For this purpose, choose *p* and *q* so that $\mathbf{x}^{(i)}\cdot\mathbf{x}^{(i)}=1$. (For vectors of dimension *n*, the generalized dot product is defined such that $\mathbf{x}\cdot\mathbf{x}=\sum_{i=1}^{n}x_i x_i$)

One may calculate explicit expressions for eigenvalues and eigenvectors for any matrix up to order (4×4), but the results are so cumbersome that, except for the (2×2) results, they are virtually useless. In practice, numerical values may be computed using several iterative techniques. Packages like Mathematica, Maple or Matlab make calculations like this easy.

The eigenvalues of a real symmetric matrix are always real, and its eigenvectors are *orthogonal*, i.e. the *i*th and *j*th eigenvectors (with $i\neq j$) satisfy $\mathbf{x}^{(i)}\cdot\mathbf{x}^{(j)}=0$.

The eigenvalues of a skew symmetric matrix are pure imaginary.

- **Spectral and singular value decomposition.** Let $[A]$ be a real symmetric $(n\times n)$ matrix. Denote the *n* (real) eigenvalues of $[A]$ by λ_i, and let $\mathbf{w}^{(i)}$ be the corresponding *normalized* eigenvectors, such that $\mathbf{w}^{(i)}\cdot\mathbf{w}^{(i)}=1$. Then, for any arbitrary vector **b**,

$$[A]\,\mathbf{b}=\sum_{i=1}^{n}\lambda_i\left(\mathbf{w}^{(i)}\cdot\mathbf{b}\right)\ \mathbf{w}^{(i)}$$

Let $[\Lambda]$ be a diagonal matrix which contains the *n* eigenvalues of $[A]$ as elements of the diagonal, and let $[Q]$ be a matrix consisting of the *n* eigenvectors as columns, i.e.

$$[\Lambda]=\begin{bmatrix}\lambda_1 & 0 & 0 & \cdots & 0\\ 0 & \lambda_2 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_n\end{bmatrix}\qquad [Q]=\begin{bmatrix}w_1^{(1)} & w_1^{(2)} & w_1^{(3)} & \cdots & w_1^{(n)}\\ w_2^{(1)} & w_2^{(2)} & w_2^{(3)} & \cdots & w_2^{(n)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ w_n^{(1)} & w_n^{(2)} & w_n^{(3)} & \cdots & w_n^{(n)}\end{bmatrix}$$

Then

$$[A]=[Q]\,[\Lambda]\,[Q]^T\qquad [Q]^T\,[Q]=[Q]\,[Q]^T=[I]\qquad [Q]^T\,[A]\,[Q]=[\Lambda]$$

Note that this gives another (generally quite useless) way to invert $[A]$

$$[A]^{-1}=[Q]\,[\Lambda]^{-1}[Q]^T$$

where $[\Lambda]^{-1}$ is easy to compute since $[\Lambda]$ is diagonal.

- **Square root of a matrix**. Let $[A]$ be a real symmetric $(n\times n)$ matrix. Denote the singular value decomposition of $[A]$ by $[A]=[Q]\,[\Lambda]\,[Q]^T$ as defined above. Suppose that $[S]=[A]^{1/2}$ denotes the square root of $[A]$, defined so that

$$[S]\,[S]=[A]$$

One way to compute $[S]$ is through the singular value decomposition of $[A]$

$$[S]=[Q]\,[\Lambda]^{1/2}[Q]^T$$

where

$$[\Lambda]^{1/2}=\begin{bmatrix}\sqrt{\lambda_1} & 0 & 0 & \cdots & 0\\ 0 & \sqrt{\lambda_2} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \sqrt{\lambda_n}\end{bmatrix}$$